Multi-Asset Options _ 2

The Margrabe Formula

The option to exchange asset S_2 for asset S_1 has payoff $(S_1(T)-S_2(T))^+$. In a complete market we know that the time 0 price of this is

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{(S_1(T)-S_2(T))^+}{e^{rT}}\right)$$

where r is the interest rate and $\mathbb{Q}$ the "risk-neutral measure", i.e., the measure under which the discounted assets are martingales. One can rewrite the payoff;

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{S_2(T)}{e^{rT}}\left(\frac{S_1(T)}{S_2(T)}-1\right)^+\right)$$

and Margrable makes two observations.

First, The process, $Y_t = \frac{S_1(t)}{S_2(t)}$ is the item of interest here and

Second, $\frac{S_2(T)}{e^{rT}} = S_2(0)\,e^{\sigma_2 W_T^2 - \sigma_2^2/2\,T}$

Which suggests a change of measure.

Taken together this amounts to viewing the value of the exchange option to be $\mathbb{E}^R((Y_T - 1)^+)$ for some measure, R. Put another way it is the Black-Scholes price of an option on an asset Y with strike equal to 1 and interest rate equal to 0.

But do we get what we expect? We look at the details.

First of all

$$Y_t = \frac{S_1(t)}{S_2(t)} = \frac{S_1(0)}{S_2(0)} e^{\sigma^1 W_t^1 - \sigma^2 W_t^2 - \frac{(\sigma^1)^2}{2}t + \frac{(\sigma^2)^2}{2}t}$$

and we use Itô's Lemma with the function $f(x) = e^x$ and the semi-martingale

$$X_t = \sigma^1 W_t^1 - \sigma^2 W_t^2 - \left(\frac{(\sigma^1)^2 - (\sigma^2)^2}{2}\right)t$$

to write down the stochastic equation for (Y_t).

$$Y_t = Y_0 + \int_0^t Y_s \, d\left(\sigma^1 W_s^1 - \sigma^2 W_s^2 - \left(\frac{(\sigma^1)^2 - (\sigma^2)^2}{2}\right)s\right) +$$

$$+ \frac{1}{2}\int_0^t Y_s \, d\langle X \rangle_s \, .$$

Now

$$\langle X \rangle_t = \langle \sigma^1 W^1 - \sigma^2 W^2 \rangle_t = \left((\sigma^1)^2 + (\sigma^2)^2 - 2\sigma^1\sigma^2\rho\right)t \, .$$

So there is some cancellation and some addition which gives

$$Y_t = 1 + \int_0^t Y_s \, d(\sigma^1 W_s^1 - \sigma^2 W_s^2) + \int_0^t (\underbrace{\sigma_2^2}_{(\sigma^2)^2} - \sigma^1\sigma^2\rho) Y_s \, ds.$$

Following up the change of measure idea, we define a new probability measure, R, equivalent to $\mathbb{Q}$ by

$$R(E) = \int_E e^{\sigma^2 W_T^2 - (\sigma_2)^2/2\, T} \, d\mathbb{Q}$$

and note that Girsanov tells us that $(W_t^2 - \sigma_2 t)$ is an R-Brownian Motion. So now,

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{S_2(T)}{e^{rT}}(Y_T - 1)^+\right) = \mathbb{E}^{\mathbb{Q}}\left(S_0^2 e^{\sigma^2 W_T^2 - \sigma_2^2 T/2}(Y_T - 1)^+\right)$$

$$= S_0^2 \mathbb{E}^R\left((Y_T - 1)^+\right).$$ So, to calculate this expectation we need to know the dynamics of Y_T under R. We already know that $(W_t^2 - \sigma^2 t)$ is an R-BM. Observe that we can rewrite the equation for Y,

$$Y_t = Y_0 + \int_0^t Y_s \, d(\sigma^1 W_s^1 - \sigma^1\sigma^2\rho s) - \int_0^t Y_s \, d(\sigma^2 W_s^2 - (\sigma^2)^2 s)$$

$$= Y_0 + \int_0^t Y_s \sigma^1 d(W_s^1 - \sigma^2\rho s) - \int_0^t Y_s \sigma^2 d(\underbrace{W_s^2 - \sigma^2 s}_{R\text{-Brownian Motion}})$$

What about the process $(W_t^1 - \sigma^2 \rho t)$? Is this too an R- Brownian Motion?

Writing, $Z_t = e^{\sigma^2 W_t^2 - \frac{(\sigma^2)^2}{2} t}$

we know that $(W_t^1 - \sigma^2 \rho t)$ will be an R- martingale iff $\left((W_t^1 - \sigma^2 \rho t) Z_t \right)$ is a Q- martingale.

Since,

$$Z_t = 1 + \int_0^t Z_s \, d\left(\sigma^2 W_s^2 - \frac{(\sigma^2)^2}{2} s\right) +$$

$$+ \frac{1}{2} \int_0^t Z_s \, d\langle \sigma^2 W^2, \sigma^2 W^2 \rangle_s$$

$$= 1 + \int_0^t \sigma^2 Z_s \, dW_s^2 \quad \text{under } Q$$

So that

$$Z_t (W_t^1 - \sigma^2 \rho t) = \int_0^t Z_s \, d(W_s^1 - \sigma^2 \rho s) +$$

$$\int_0^t (W_s^1 - \sigma^2 \rho s) \sigma^2 Z_s \, dW_s^2 +$$

$$\langle W_t^1, \int_0^t \sigma^2 Z_s \, dW_s^2 \rangle$$

$$= \int_0^t Z_s \, dW_s^1 - \int_0^t \sigma^2 Z_s \rho \, ds +$$

$$\int_0^t (W_s^1 - \sigma^2 \rho s) \sigma^2 Z_s \, dW_s^2 + \int_0^t \sigma^2 Z_s \rho \, ds$$

$$= \int_0^t Z_s \, dW_s^1 + \int_0^t (W_s^1 - \sigma^2 \rho s) \sigma^2 Z_s \, dW_s^2$$

which is a $\mathbb{Q}$-martingale. So $(W_t^1 - \sigma^2 \rho t)$ is a cts R-martingale. Now if we can show that the R-quadratic variation of $(W_t^1 - \sigma^2 \rho t)$ is actually t this this too will be an R-Brownian Motion. We use the characterisation of quadratic variation: "$\langle X \rangle$ is the (unique) natural bounded variation process such that $X^2 - \langle X \rangle$ is an L^1-martingale." So we would want $(W_t^1 - \sigma^2 \rho t)^2 - t$ an R-martingale. Or, equivalently, $(((W_t^1 - \sigma^2 \rho t)^2 - t) Z_t)$ is a $\mathbb{Q}$-martingale. Let us see.

$$(W_t^1 - \sigma^2 \rho t)^2 - t = (W_t^1)^2 - 2\sigma^2 \rho t W_t^1 + (\sigma^2)^2 \rho^2 t^2 - t$$

$$= 2\int_0^t W_s^1 \, dW_s^1 - 2\sigma^2 \rho \left(\int_0^t W_s^1 \, ds + \int_0^t s \, dW_s^1 \right) +$$

$2(\sigma^2)^2 \rho^2 \int_0^t s \, ds$, by Itô's Lemma. Now we use the product rule

$$((W_t^1 - \sigma^2 \rho t)^2 - t) Z_t =$$

$$\int_0^t Z_s W_s^1 \, dW_s^1 - 2\sigma^2 \rho \int_0^t Z_s W_s^1 \, ds - 2\sigma^2 \rho \int_0^t s Z_s \, dW_s^1 +$$

$$2(\sigma^2)^2 \rho^2 \int_0^t s Z_s \, ds + \int_0^t \sigma^2 ((W_s^1 - \sigma^2 \rho s)^2 - s) Z_s \, dW_s^2 +$$

$$\left\langle \int_0^t \sigma^2 Z_s \, dW_s^2 \, , \, 2\int_0^t W_s^1 \, dW_s^1 - 2\sigma^2 \rho \int_0^t s \, dW_s^1 \right\rangle .$$

The quadratic variation term is

$$\left\langle \int_0^t \sigma^2 Z_s\, dW_s^2,\ 2\int_0^t W_s^1\, dW_s^1 \right\rangle = \int_0^t 2\sigma^2 W_s^1 Z_s \rho\, ds$$

$$\left\langle \int_0^t \sigma^2 Z_s\, dW_s^2,\ -2\sigma^2\rho \int_0^t s\, dW_s^1 \right\rangle = -\int_0^t 2(\sigma^2)^2 s Z_s \rho^2\, ds$$

These terms cancel with the other "ds" terms leaving

$$\int_0^t Z_s W_s^1\, dW_s^1 - 2\sigma^2\rho \int_0^t s Z_s\, dW_s^1 + \int_0^t \sigma^2 Z_s\left((W_s^1 - \sigma^2\rho s)^2 - s\right) dW_s^2$$

which is a $\mathbb{Q}$-martingale. So "t" is the R-quadratic variation of $(W_t^1 - \sigma^2\rho t)$ and this is an R-Brownian Motion.

We note, further, that under $\mathbb{Q}$,

$$(W_t^1 - \sigma^2\rho t)(W_t^2 - \sigma^2 t) = \underbrace{\int_0^t (W_s^1 - \sigma^2\rho s)\, d(W_s^2 - \sigma^2 s)}_{N_1(t)}$$

$$+ \underbrace{\int_0^t (W_s^2 - \sigma^2 s)\, d(W_s^1 - \sigma^2\rho s)}_{N_2(t)} + \rho t$$

So that

$(W_t^1 - \sigma^2\rho t)(W_t^2 - \sigma^2 t) - \rho t$ is the sum of the two integrals above. Using the product rule.

$$\left((W_t^1 - \sigma^2\rho t)(W_t^2 - \sigma^2 t) - \rho t\right) Z_t =$$

$$\int_0^t Z_s (W_s^1 - \sigma^2\rho s)\, d(W_s^2 - \sigma^2 s) + \int_0^t Z_s (W_s^2 - \sigma^2 s)\, d(W_s^1 - \sigma^2\rho s)$$

$$+ \int_0^t (N_1(s) + N_2(s))\, \sigma^2 Z_s\, dW_s^2 \qquad +$$

$$\left\langle \int_0^t (W_s^1 - \sigma^2\rho s)\, dW_s^2 + \int_0^t (W_s^2 - \sigma^2 s)\, dW_s^1,\ \int_0^t \sigma^2 Z_s\, dW_s^2 \right\rangle_t$$

The quadratic variation term is

$$\int_0^t \sigma^2 W_s^1 Z_s ds - \int_0^t (\sigma^2)^2 \rho s Z_s ds + \int_0^t \sigma^2 W_s^2 Z_s \rho ds - \int_0^t (\sigma^2)^2 s Z_s \rho ds$$

and these terms cancel with the "deterministic terms" in N_1 and N_2 to leave only the terms,

$$\int_0^t Z_s (W_s^1 - \sigma^2 \rho s) dW_s^2 + \int_0^t Z_s (W_s^2 - \sigma^2 s) dW_s^1 +$$

$$\int_0^t (N_1(s) + N_2(s)) \sigma^2 Z_s dW_s^2$$

which is a Q_1-martingale. So the R-cross variation of $(W_t^1 - \rho\sigma^2 t)$ and $(W^2 - \sigma^2 t)$ is ρt.

We know then that under R,

$$Y_t = Y_0 + \int_0^t Y_s \sigma^1 dW_s^1(R) - \int_0^t \sigma^2 Y_s dW_s^2(R)$$

where $\langle W^1(R), W^2(R) \rangle_t = \rho t$. Noting that

$$\langle \sigma^1 W^1(R) - \sigma^2 W^2(R) \rangle = \left((\sigma^1)^2 + (\sigma^2)^2 - 2\sigma^1\sigma^2\rho\right)t$$

we see that,

$$W^R = \frac{\sigma^1 W^1(R) - \sigma^2 W^2(R)}{\left((\sigma^1)^2 + (\sigma^2)^2 - 2\sigma^1\sigma^2\rho\right)^{1/2}}$$ is an R-

Brownian Motion, and

$$Y_t = Y_0 + \int_0^t Y_s \sigma dW_s^R$$

where $\sigma = \left((\sigma^1)^2 + (\sigma^2)^2 - 2\sigma^1\sigma^2\rho\right)^{1/2}$.

So,

$$Y_t = Y_0 e^{\sigma W_t^R - \frac{\sigma^2}{2}t}.$$

So when we calculate

$$\mathbb{E}^{Q}\left((S_T^1 - S_T^2)^+ e^{-rT}\right) = \mathbb{E}^{R}\left(S_0^2 (Y_T - 1)^+\right)$$

we know that the last term is

$$S_0^1 N(d_1) - S_0^2 N(d_2) \quad \left(\text{B\&S with } r=0,\ K=1,\ Y_0 = \frac{S_0^1}{S_0^2}\right)$$

where

$$d_1 = \frac{\log\left(\frac{S_0^1}{S_0^2}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and $\sigma = (\sigma^1)^2 + (\sigma^2)^2 - 2\sigma^1\sigma^2\rho$.

Hedging The Exchange Option:

Cross Currency Options

Consider an asset S whose value is denominated in a foreign currency, F. We are interested in an option on S which pays off in our own domestic currency, D.

Domestic	f_t	Foreign
$f_t S_t$		S_t
interest, r_D		r_F interest,
$P_D(t,T) = e^{-r_D(T-t)}$		$P_F(t,T) = e^{-r_F(T-t)}$
$f_t P_F(t,T)$		$P_F(t,T)$
$P_D(t,T)$		$\dfrac{P_D(t,T)}{f_t}$

Here f_t is the exchange rate where f_t Domestic $=$ 1 Foreign.

Some remarks: Each of P_D and P_F deliver one unit of currency at time T. Suppose that we arrange at time $t=0$ to deliver 1 unit of currency F at time T. Then at time $t=0$ we require $P_F(0,T)$ of the foreign bond or its equivalent, $f_0 P_D(0,T)$ of domestic bond (so we can buy $P_F(0,T)$). If we borrow $f_0 P_D(0,T)$ at $t=0$ then at time T we would owe $e^{r_D T} f_0 P_F(0,T) = f_0 e^{(r_D - r_F)T} = f_0 \frac{P_F(0,T)}{P_D(0,T)}$.

Assuming our domestic economy has no arbitrage this amount must agree with the time T value of 1 unit of foreign currency. So the forward price of 1 unit of F at time T is $f_0 e^{(r_D - r_F)T}$. This is redolent of a domestic asset paying dividends at rate r_F ....

In the domestic economy the traded assets are, with some contraction of notation,

$$Z_t = P_D(t,T)$$

$$Y_t = f_t P_F(t,T)$$

and under the Domestic Risk-Neutral Measure the (foreign) price of S is

$$S_t = S_0 + \int_0^t S_s \mu_S ds + \int_0^t S_s \sigma_S dW_s^S$$

and we assume S pays dividends at a continuous rate, q. We write

$$X_t = f_t S_t$$

for the domestic value of S. In addition to this we assume the exchange rate is log-normal too,

$$F_t = F_0 + \int_0^t \mu_F F_s \, ds + \int_0^t F_s \sigma^f dW_s^f$$

where $\langle W^S, W^F \rangle_t = \rho t$.

Under the D.R.M. the discounted value of assets should be martingale, so the discounted (domestic) value of the foreign bond will be a martingale, that is,

$$e^{-r_D t} F_t P_F(t,T) = e^{-r_F T} F_t e^{-(r_D - r_F)t}$$

is a martingale. The product rule tells us,

$$e^{-(r_D-r_F)t} F_t = F_0 + \int_0^t e^{-(r_D-r_F)s} \cdot \mu_F F_s \, ds + \int_0^t e^{-(r_D-r_F)s} F_s \sigma^f dW_s^f$$
$$- \int_0^t F_s e^{-(r_D-r_F)s} (r_D - r_F) \, ds$$

$$= F_0 + \int_0^t (\mu_F - (r_D - r_F)) e^{-(r_D-r_F)s} F_s \, ds + \int_0^t e^{-(r_D-r_F)s} F_s \sigma^f dW_s^f.$$

So we must have $\mu_F = r_D - r_F$.

If we form a portfolio of foreign assets comprising ϕ_t of S_t with the remainder in P_F. The portfolio value, V_t, is

$$V_t = V_0 + \int_0^t \phi_s dS_s + \int_0^t \frac{(V_s - \phi_s S_s)}{P_F(s,T)} dP_F(s,T) + \int_0^t q\phi_s S_s ds$$

$$= V_0 + \int_0^t \phi_s dS_s + \int_0^t (V_s - \phi_s S_s) r_F ds + \int_0^t q\phi_s S_s ds$$

$$= V_0 + \int_0^t V_s r_F ds + \int_0^t \phi_s S_s (\mu^S + q - r_F) ds + \int_0^t \phi_s S_s \sigma^S dW_s^S$$

The domestic value of this portfolio is $U_t = f_t V_t$ and an application of the product rule gives

S k i p

$$U_t = f_0 V_0 + \int_0^t V_s f_s \mu_f ds + \int_0^t V_s f_s \mu_f dW_s^f + \int_0^t f_s V_s r_F ds + \int_0^t f_s \phi_s S_s (\mu^S + q - r_F) ds + \int_0^t f_s \phi_s S_s \sigma^S dW_s^S + \int_0^t f_s \phi_s S_s \sigma^f \sigma^S \rho ds$$

since $\mu_f = r_D - r_F$ and $U_s = V_s f_s$

$$U_t = U_0 + \int_0^t U_s r_D ds + \int_0^t \sigma_F U_s dW_s^f + \int_0^t \phi_s f_s S_s \sigma^S dW_s^S + \int_0^t \phi_s f_s S_s (\mu^S + q - r_F + \rho\sigma^S \sigma^f) ds$$

The discounted value of this (domestic) portfolio is (product rule)

$$e^{-r_D t} U_t = V_0 + \int_0^t e^{-r_D s} U_s \sigma^f dW_s^f + \int_0^t e^{-r_D s} \phi_s f_s S_s \sigma^S dW_s^S + \int_0^t e^{-r_D s} \phi_s f_s S_s (\mu^S + q - r_F + \rho\sigma^S \sigma^F) ds .$$

For this to be a martingale (as it must be) we must have

$$\mu^s + q - r_F + \sigma^f \sigma^s \rho = 0$$

i.e. $$\mu^s = r_F - q - \sigma^f \sigma^s \rho$$

So under DRM

$$F_t = F_0 + \int_0^t F_s (r_D - r_F) ds + \int_0^t \sigma^f F_s dW_s^f$$

$$S_t = S_0 + \int_0^t S_s (r_F - q - \rho \sigma^f \sigma^s) ds + \int_0^t S_s \sigma^s dW_s^s .$$

Now we can compute $X_t = F_t S_t$;
Exercise: use the product rule to show

$$X_t = X_0 + \int_0^t X_s (r_D - q) ds + \int_0^t X_s d(\sigma^s W_s^s + \sigma^f W_s^f).$$

Using a familiar trick,

$$\frac{\sigma^s W^s + \sigma^f W^f}{((\sigma^s)^2 + (\sigma^f)^2 + 2\sigma^f \sigma^s \rho)^{1/2}} = W$$

is a BM under DRN. So we can rewrite X as

$$X_t = X_0 + \int_0^t X_s (r_D - q) ds + \int_0^t X_s \sigma dW_s$$

where $\sigma = ((\sigma^s)^2 + (\sigma^f)^2 + 2\sigma^f \sigma^s \rho)^{1/2}$

Now we can discuss options on the foreign asset S. For example a standard call, $(X_T - K)^+$. From what we have written above it is clear that this is valued as a normal 'B&S' option with the volatility $\sigma = \left((\sigma^S)^2 + (\sigma^f)^2 + 2\sigma^f\sigma^S\rho\right)^{1/2}$.

<u>Currency Protected Options</u>: The option above has value which incorporates the fluctuations arising from the exchange rate. One can arrange for an agreed fixed exchange rate to convert the payoff into domestic currency. More precisely, the payoff is set to be

$$A_0 (S_T - K)^+$$

where A_0 is an exchange rate agreed at the outset of the contract. In the domestic economy the time $t=0$ value is

$$A_0 e^{-r_D T} \mathbb{E}\left((S_T - K)^+\right)$$

where $\mathbb{E}$ is the expectation with respect to the DRM. Recall that under DRM,

$$S_t = S_0 + \int_0^t S_s (r_F - q - \rho\sigma^S\sigma^f)\,ds + \int_0^t S_s \sigma^S dW_s^S.$$

One can rewrite the drift :

$$r_F - q - \rho\sigma^S\sigma^f = r_D - (q + r_D - r_F + \rho\sigma^S\sigma^f)$$

and this gives two 'views' for valuation:

We have $S_T = S_0 e^{\sigma^S W_T^S + (r_F - q - \sigma^S \sigma^f \rho - (\sigma^S)^2/2)T}$
so that

$$\mathbb{E}((S_T - K)^+) = \mathbb{E}(S_T I_{\{S_T > K\}}) - K\mathbb{E}(I_{\{S_T > K\}})$$

while

$$S_T > K \iff W_T^S > \frac{\log\left(\frac{K}{S_0}\right) - \left(r_F - q - \sigma^S \sigma^f \rho - \frac{(\sigma^S)^2}{2}\right)T}{\sigma^S}$$

i.e.

$$N(0,1) < \frac{\log\left(\frac{S_0 e^{(r_F - q - \rho\sigma^S\sigma^f)T}}{K}\right) - \frac{(\sigma^S)^2}{2}T}{\sigma^S\sqrt{T}}$$

and one identifies $S_0 e^{(r_F - q - \rho\sigma^S\sigma^f)T}$ with the forward price, by analogy with the B&S case where we can write

$$C(0,S_0) = e^{-rT}(F N(d_1) - K N(d_2))$$

with $d_1 = \dfrac{\log\left(\frac{F}{K}\right) + \sigma^2/2\, T}{\sigma\sqrt{T}}$ ......

you can finish the details.

Alternatively we can think in terms of the usual dividend paying stock and regard S under DRM as

$$S_t = S_0 + \int_0^t S_s\left(r_D - (q + r_D - r_F + \sigma^S\sigma^f\rho)\right)ds + \int_0^t \sigma^S S_s\, dW_s^S$$

so it has a dividend rate equal to

$q + r_D - r_F + \rho\sigma^S\sigma^f$. Refer to the Chapter on dividends paying assets for the option pricing formula.

Hedging The Option: In order to arrive at a hedge for this option we revert to the B&S approach. However to do this we work in the objective measure. Our assumption under the DRM are that both S and f are log-normal and, therefore, X and f are log-normal. If we change from DRM back to the objective measure we know, by now, that this will amount to changing the <u>drift</u> of X and f rather than the nature of the process driving X and f, they will still have the <u>form</u>

$$X_t = X_0 + \int_0^t \lambda X_s\,ds + \int_0^t \tilde{\sigma} X_s\,d\tilde{W}_s \quad *$$

$$f_t = f_0 + \int_0^t \nu f_s\,ds + \int_0^t \sigma^f f_s\,dW^f_s$$

for some drifts, λ and ν. We can also assume that $\tilde{W}$ and W^f will have the same correlation as before. To hedge the option we form a portfolio long the option and short in X, Y, Z. If we assume that the option value is given by $C(t, S_t)$, where $C(t,x)$ is some (nice) $C^{1,2}$ function of t and x, then we can

$*$: see later

write $g(t,x,f) = C(t,\frac{x}{f})$ and our portfolio has value,

$$V_t = g(t, X_t, f_t) - \phi_t X_t - \psi_t Y_t - \chi_t Z_t - \int_0^t \phi_s X_s q_v \, ds$$

(recall that S pays a divvy...). Following the B&S method we will use Itô's Lemma on g and the self-financing property on the portfolio to obtain an expression for V_t. We want $V_t \equiv 0$. As with B&S previously, we choose ϕ, ψ, χ in stages to achieve this. First of all we need some preparatory detail.

(i)

Our assumption about X includes the construction of $\tilde{W}$ as

$$\tilde{W} = \frac{\sigma^S W^S + \sigma^f W^f}{\tilde{\sigma}}$$

where $\tilde{\sigma} = \left((\sigma^S)^2 + (\sigma^f)^2 + 2\sigma^S\sigma^f\rho\right)^{1/2}$

(ii) We are going to use Itô's Lemma with $g(t,x,f)$ so it may be helpful to compute some partial derivatives:

$$g(t,x,f) = C(t,\frac{x}{f})$$

So $\frac{\partial g}{\partial t} = \frac{\partial C}{\partial t}$, $\frac{\partial g}{\partial x} = \frac{1}{f}\frac{\partial C}{\partial x}$, $\frac{\partial g}{\partial f} = \frac{-x}{f^2}\frac{\partial C}{\partial x}$

$$\frac{\partial^2 g}{\partial x^2} = \frac{1}{F^2}\frac{\partial^2 c}{\partial x^2}, \quad \frac{\partial^2 g}{\partial f^2} = \frac{2x}{F^3}\frac{\partial c}{\partial x} + \frac{x^2}{F^4}\frac{\partial^2 c}{\partial x^2}$$

$$\frac{\partial^2 g}{\partial x \partial f} = \left(\frac{-1}{F^2}\right)\frac{\partial c}{\partial x} - \left(\frac{x}{F^2}\right)\frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}(t, x/f)\right)$$

$$= \frac{-1}{F^2}\frac{\partial c}{\partial x} - \frac{x}{F^3}\frac{\partial^2 c}{\partial x^2}$$

Doing Itô on g....

$$g(t,X_t,f_t) = g(0,X_0,f_0) + \int_0^t \frac{\partial g}{\partial t}(s,X_s,f_s)\,ds +$$

$$+ \int_0^t \frac{\partial g}{\partial x}(s,X_s,F_s)\,dX_s + \int_0^t \frac{\partial g}{\partial f}(s,X_s,f_s)\,df_s +$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}\,d\langle X,X\rangle_s + \int_0^t \frac{\partial^2 g}{\partial x \partial f}\,d\langle X,f\rangle_s + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial f^2}\,d\langle f,f\rangle_s$$

S
K
I
P
T
O

There are a lot of cross-variations to calculate.

$$\frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}\,d\langle X,X\rangle = \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}\,\tilde{\sigma}^2 X^2\,ds$$

$$= \frac{1}{2}\int_0^t \frac{\partial^2 c}{\partial x^2}\,\tilde{\sigma}^2 \frac{X^2}{F^2}\,ds \quad (1).$$

$$\int_0^t \frac{\partial^2 g}{\partial x \partial f}\,d\langle X,f\rangle = \int_0^t \frac{\partial^2 g}{\partial x \partial f}\,\tilde{\sigma}\sigma^f fX\,d\langle \tilde{W}, W^f\rangle$$

(i)&(ii) $$= \int_0^t \left(\cancel{\frac{-1}{F^2}\frac{\partial c}{\partial x}} - \frac{X}{F^3}\frac{\partial^2 c}{\partial x^2}\right)\tilde{\sigma}\sigma^f fX\left(\frac{\sigma^s}{\tilde{\sigma}}\rho + \frac{\sigma^f}{\tilde{\sigma}}\right)ds$$

$$= \int_0^t \left(\cancel{-\frac{1}{F^2}\frac{\partial C}{\partial x}} - \cancel{\frac{X}{F^3}\frac{\partial^2 C}{\partial x^2}} \right)\left(\sigma^s \sigma^f \rho + \cancel{(\sigma^f)^2}\right) X \cancel{f} \, ds \quad ②.$$

$$\frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial F^2} d\langle F, F\rangle = \frac{1}{2}\int_0^t \left(\cancel{\frac{2X}{F^3}\frac{\partial C}{\partial x}} + \cancel{\frac{X^2}{F^4}\frac{\partial^2 C}{\partial x^2}} \right)\cancel{(\sigma^f)^2} \cancel{F^2} \, ds \quad ③.$$

There are some cancellations; indicated. The yellow yeilds a remainder while the red yeilds zero. So ② and ③ leave us with

$$\int_0^t -\cancel{\left(\frac{X}{F}\right)^2 \sigma^f \sigma^s \rho \frac{\partial^2 C}{\partial x^2}} \, ds + \int_0^t -\left(\frac{X}{F}\right) \sigma^f \sigma^s \rho \frac{\partial C}{\partial x} \, ds +$$

$$-\frac{1}{2}\int_0^t \cancel{\left(\frac{X}{F}\right)^2 (\sigma^f)^2 \frac{\partial^2 C}{\partial x^2}} \, ds$$

but at the same time $(\tilde{\sigma})^2 = (\sigma^s)^2 + (\sigma^f)^2 + 2\rho\sigma^f\sigma^s$
so the term ① becomes

$$\frac{1}{2}\int_0^t \left(\frac{\partial^2 C}{\partial x^2}\left(\frac{X}{F}\right)^2 (\sigma^s)^2 + \cancel{\frac{\partial^2 C}{\partial x^2}\left(\frac{X}{F}\right)^2 (\sigma^f)^2} + \cancel{2\rho\sigma^f\sigma^s \left(\frac{X}{F}\right)^2 \frac{\partial^2 C}{\partial x^2}} \right) ds$$

and there are more cancellations some giving zero some not; we are left with

$$-\int_0^t \left(\frac{X}{F}\right)^2 \sigma^f \sigma^s \rho \frac{\partial C}{\partial x} \, ds + \frac{1}{2}\int_0^t \left(\frac{X}{F}\right)^2 \frac{\partial^2 C}{\partial x^2} (\sigma^s)^2 \, ds$$

Our expression for $g(t, X_t, F_t)$ becomes

$$g(t, X_t, F_t) = g(0, X_0, F_0) + \int_0^t \frac{\partial C}{\partial t} ds + \int_0^t \frac{1}{F}\frac{\partial C}{\partial x} dX$$

$$- \int_0^t \left(\frac{X}{F^2}\right)\frac{\partial C}{\partial x} dF - \int_0^t \left(\frac{X}{F}\right) \sigma^f \sigma^s \rho \frac{\partial C}{\partial x} ds + \frac{1}{2}\int_0^t \left(\frac{X}{F}\right)^2 (\sigma^s)^2 \frac{\partial^2 C}{\partial x^2} ds.$$

We subtract from this our (self-financing) portfolio whose value is

$$P_0 + \int_0^t \phi_s dX_s + \int_0^t \psi_s dY_s + \int_0^t \chi_s dZ_s + \int_0^t q\phi_s X_s ds$$

Now, $Y_t = f_t P_F(t,T) = f_t e^{-r_F(T-t)}$

$$= Y_0 + \int_0^t (\nu + r_F) Y_s ds + \int_0^t \sigma^f Y_s dW_s^f$$

while

$$-\int_0^t \left(\frac{X}{f^2}\right) \frac{\partial C}{\partial x} df_s = -\int_0^t \left(\frac{X}{f^2}\right) \frac{\partial C}{\partial x} \nu f ds - \int_0^t \left(\frac{X}{f^2}\right) \frac{\partial C}{\partial x} \sigma^f f dW_s^f$$

subtracting $\int_0^t \psi_s dY_s$ gives

$$-\int_0^t \left(\frac{X}{f^2}\right) \frac{\partial C}{\partial x} df_s - \int_0^t \psi_s dY_s =$$

$$-\int_0^t \left((\nu + r_F) Y\psi + \left(\frac{X}{f}\right) \nu \frac{\partial C}{\partial x} \right) ds - \int_0^t \left(\psi \sigma^f Y + \left(\frac{X}{f}\right) \sigma^f \frac{\partial C}{\partial x} \right) dW_s^f$$

So

$$V_t = g(0, x_0, f_0) + \int_0^t \frac{\partial C}{\partial t} ds + \int_0^t \left(\frac{1}{f} \frac{\partial C}{\partial x} - \phi \right) dX$$

$$- \int_0^t \left((\nu + r_F) Y\psi + \left(\frac{X}{f}\right) \nu \frac{\partial C}{\partial x} \right) ds$$

$$- \int_0^t \left(\psi \sigma^f Y + \left(\frac{X}{f}\right) \sigma^f \frac{\partial C}{\partial x} \right) dW^f - \int_0^t \left(\frac{X}{f}\right) \sigma^f \sigma^S \rho \frac{\partial C}{\partial x} ds$$

$$+ \frac{1}{2} \int_0^t \left(\frac{X}{f}\right)^2 (\sigma^S)^2 \frac{\partial^2 C}{\partial x^2} ds - (\phi_0 X_0 + \psi_0 Y_0 + \chi_0 Z_0)$$

$$- \int_0^t q\phi_s X_s ds - \int_0^t r_D \chi_s Z_s ds$$

We observe that setting $\phi = \frac{1}{F}\frac{\partial C}{\partial x}$ and $\psi = -\frac{S}{Y}\frac{\partial C}{\partial x}$ removes the stochastic integral terms and if we write $S = X/F$ (as it is) then

$$V_t = V_0 + \int_0^t \left(\frac{\partial C}{\partial t} + \tfrac{1}{2}(\sigma^S)^2 S^2 \frac{\partial^2 C}{\partial x^2} + (r_F - q - \rho\sigma^f\sigma^S) S \frac{\partial C}{\partial x} - \chi_s r_D Z_s \right) ds$$

Chi

Since we have eliminated the stochastic nature of V by our choice of ϕ and ψ then V must satisfy the deterministic equation

$$V_t = V_0 + \int_0^t r_D V_s \, ds$$

for otherwise there would be arbitrage.* Coupling this with our equation for V, preceeding the last equation, and noting

$$V_s = (C_s - \phi_s X_s - \psi_s Y_s - \chi_s Z_s)$$

$$= \left(C_s - \frac{1}{F}\frac{\partial C}{\partial x} X_s + \frac{S}{Y_s}\frac{\partial C}{\partial x} Y_s - \chi_s Z_s\right)$$

$$= (C_s - \chi_s Z_s)$$

since $\frac{X}{F} = S$, then we get

* This is 'obvious' financially but less so mathematically.

$$0 = V_0 + \int_0^t \left(\frac{\partial C}{\partial t} + \frac{1}{2}(\sigma^S)^2 S^2 \frac{\partial^2 C}{\partial x^2} + (r_F - q - \rho\sigma^S\sigma^f) S \frac{\partial C}{\partial x} - r_D C \right) ds$$

And by an argument we have rehearsed earlier for the B&S pde. we must have $V_0 = C(0, S_0) - \phi_0 X_0 - \psi_0 Y_0 - x_0 Z_0 = 0$ and $C(t,x)$ should satisfy

$$0 = \frac{\partial C}{\partial t} + \frac{1}{2}(\sigma^S)^2 x^2 \frac{\partial^2 C}{\partial x^2} + (r_F - q - \rho\sigma^S\sigma^f) x \frac{\partial C}{\partial x} - r_D C$$

with appropriate boundary conditions. The most important of these is

$$C(T, x) = A_0 (x - K)^+ .$$

If we write

$$r_F - q - \rho\sigma^S\sigma^f = r_D - (q + r_D - r_F + \rho\sigma^S\sigma^f)$$

then we can view $C(t,x)$ as the solution to A_0 times the B&S p.d.e. with risk-less rate r_D, volatility σ^S and dividend rate

$$q + r_D - r_F + \rho\sigma^S\sigma^f .$$

Compare with the 'risk-neutral' valuation.

To see that the portfolio replicates we define,

$$x_t = C(t, S_t) / Z_t$$

because then $V_s - x_s Z_s = 0$.
To see that the portfolio is self-financing we note that with x chosen as above $V_t \equiv 0$
so, writing $W_t = \phi_t X_t + \psi_t Y_t + x_t Z_t = C(t, S_t)$

$$W_t = C(0,S_0) + \int_0^t \frac{\partial C}{\partial t}(s,S_s)ds + \int_0^t \frac{\partial C}{\partial x}(s,S_s)dS_s +$$

$$\frac{1}{2}\int_0^t \frac{\partial^2 C}{\partial x^2}(s,S_s)(\sigma^S)^2 S_s^2 ds$$

Since C satisfies the pde above

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}(\sigma^S)^2 x^2 \frac{\partial^2 C}{\partial x} = r_D C - (r_F - q - \rho\sigma^f\sigma^S) x \frac{\partial C}{\partial x}$$

so

$$W_t = C(0,S_0) + \int_0^t r_D C(s,S_s)ds + \int_0^t \frac{\partial C}{\partial x}(s,S_s)dS_s$$

$$- \int_0^t (r_F - q - \rho\sigma^f\sigma^S) S_s \frac{\partial C}{\partial x}(s,S_s)\, ds$$

Since $\frac{X}{F} = S$ then

$$S_t = \frac{X_0}{F_0} + \int_0^t \frac{1}{F_s}dX_s - \int_0^t \frac{S_s}{F_s}dF_s - \int_0^t (\rho\sigma^S\sigma^f S_s)ds$$

and this identifies "dS" for us:

$$W_t = C(0,S_0) + \int_0^t r_D C(s,S_s)\,ds + \int_0^t \frac{\partial C}{\partial x}(s,S_s)\frac{1}{F_s}\,dX_s$$

$$- \int_0^t \frac{\partial C}{\partial x}(s,S_s)\frac{S_s}{F_s}\,df_s - \int_0^t \frac{\partial C}{\partial x}(s,S_s)\rho\sigma^S\sigma^f S_s\,ds$$

$$- \int_0^t (r_F - q - \rho\sigma^f\sigma^S) S_s \frac{\partial C}{\partial x}(s,S_s)\,ds .$$

$$= C(0,S_0) + \int_0^t r_D C(s,S_s)\,ds + \int_0^t \frac{\partial C}{\partial x}(s,S_s)\frac{1}{F_s}\,dX_s$$

$$+ \int_0^t q S_s \frac{\partial C}{\partial x}(s,S_s)\,ds$$

$$- \int_0^t r_F S_s \frac{\partial C}{\partial x}(s,S_s)\,ds - \int_0^t \frac{\partial C}{\partial x}(s,S_s)\frac{S_s}{F_s}\,dF_s .$$

Now, $\int_0^t \phi_s\,dX_s = \int_0^t \frac{\partial C}{\partial x}(s,S_s)\frac{1}{F_s}\,dX_s$

$$\int_0^t x_s\,dZ_s = \int_0^t x_s Z_s r_D\,ds = \int_0^t r_D C(s,S_s)\,ds$$

$$\int_0^t q S_s \frac{\partial C}{\partial x}(s,S_s)\,ds = \int_0^t q\phi_s X_s\,ds$$

Now $Y_t = e^{-r_F(T-t)} F_t$

$$= e^{-r_F T} F_0 + \int_0^t r_F Y_s\,ds + \int_0^t e^{-r_F(T-s)}\,dF_s$$

$$= e^{-r_F T} f_0 + \int_0^t r_F Y_s\, ds + \int_0^t Y_s \frac{1}{f_s} df_s$$

which shows us that "dY" is hiding in the term

$$-\int_0^t r_F S_s \frac{\partial C}{\partial x}(s, S_s)\, ds - \int_0^t \frac{\partial C}{\partial x}(s,S_s) \frac{S_s}{f_s} df_s$$

$$\left(\psi = -\frac{S}{Y} \frac{\partial C}{\partial x} \right)$$

$$= \int_0^t -\frac{S_s}{Y_s}\frac{\partial C}{\partial x}(s,S_s) \cdot Y_s\, r_F\, ds + \int_0^t -\frac{S_s}{Y_s}\frac{\partial C}{\partial x}(s,S_s) \frac{Y_s}{f_s} df_s$$

$$= \int_0^t \psi_s Y_s r_F\, ds + \int_0^t \psi_s \frac{Y_s}{f_s} df_s$$

$$= \int_0^t \psi_s\, dY_s$$

So the portfolio is self-financing.

The hedging strategy is (ϕ, ψ, χ). Observe that

$$\phi_t X_t + \psi_t Y_t = \frac{1}{f_t}\frac{\partial C}{\partial x} X_t - \frac{S_t}{Y_t}\frac{\partial C}{\partial x} Y_t$$

$$= 0 \quad \text{since} \quad \frac{X_t}{f_t} = S_t$$

while $\chi_t Z_t = C(t, S_t)$. The first two terms eliminate the FX risk while the domestic bond value tracks the option. The strategy ϕ is a 'delta hedge' in the foreign currency financed by foreign currency borrowing — which is the strategy ψ. Changes in the hedge are diverted to the domestic riskless bond as they accrue.